

Some properties of unstable monopoles in Super-QCD

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We study embeddings of the Prasad-Sommerfield monopole solution in $SU(N_c)$ Super-QCD ($N_c \geq 3$), where the role of the Higgs field is played by the squarks in the fundamental representation. Classically, the resulting configurations live in a phase with unbroken $SU(k)$ subgroups of $SU(N_c)$ (as a result they are not topologically stable). The structure of zero modes of these monopoles is such that they can be naturally interpreted as massive chiral superfields, with R charge one and baryon number zero. They transform in the adjoint representation of a dual gauge group defined using the Goddard-Nuyts-Olive (GNO) framework. We discuss the possible applications of these monopoles to $N = 1$ duality, and more generally the possibility of relating GNO type dual gauge groups to those appearing in $N=1$ duality.

1. Introduction

Considerable evidence has been found for a version of electric-magnetic duality that governs the low energy dynamics of certain $N = 1$ supersymmetric theories [1], [2]. Among the more striking aspects is the statement that in the infrared $SU(N_c)$ theory with N_f flavours ($N_c + 2 \leq N_f \leq 3N_c$) is an interacting conformal field theory which has a dual description in terms of magnetic variables with gauge group $SU(N_f - N_c)$ containing N_f quarks and N_f^2 mesons. This differs from previous electric-magnetic duality conjectures (mainly in the context of extended $N = 2$ and $N = 4$ SUSY theories) in various aspects. Since the low energy theory has a manifest non-abelian gauge symmetry, the simple abelian duality transformations of Maxwell theory cannot be used, even at a formal level, to understand the operator mapping. In the $N = 2$ and $N = 4$ dualities [3,4,5] monopoles [6][7], play a fundamental role, being the elementary degrees of freedom in the dual formulations. In these theories the relevant monopoles saturate a BPS bound and live in small representations [8], so although the semiclassical construction of states starting from non-trivial solutions is only valid, in the region of large Higgs vevs, these states can be argued to survive in the strong coupling region. In $N = 1$ theories in $4D$, massive states cannot be protected by such a BPS saturation, so there is no guarantee that states found by semiclassical treatments will survive in the strong coupling region. In these respects $N = 1$ duality is perhaps more mysterious than other field theoretic dualities proposed so far. For some special $N = 1$ theories, related to theories with higher supersymmetry or having an abelian Coulomb phase, duality can be related to monopole physics [9]. One may naturally ask to what extent these relations survive in the $SU(N_c)$ examples. The first problem one meets is that super-QCD does not seem to support standard monopole configurations of 't Hooft-Polyakov type, which are constructed using adjoint Higgs fields, so there are not even natural candidates in the semiclassical region.

In this paper we describe a class of non-singular finite energy monopole solutions of such theories. Like any classical monopoles living in a phase with unbroken $SU(k)$ groups they are not topologically stable. They have an interesting zero mode structure and admit an interpretation as an adjoint chiral superfield. The interpretation of the gauge quantum numbers is done using the framework of Goddard, Nuyts and Olive [10] (see also [11]), with the additional feature that we discuss and use the freedom of associating a dual group to a sublattice of the lattice of magnetic weights allowed by Dirac quantization. We comment more generally on the possible applications of this freedom in $N = 1$ duality. Although it

is conceivable that no vestige of $N = 1$ duality in the $SU(N)$ models can be captured with semiclassical methods, we speculate on possible scenarios where these monopole solutions would be related to duality.

2. Construction of Classical Solutions

We construct a class of monopole solutions of $SU(N_c)$ ($N_c > 3$) Yang Mills theories coupled to fundamental matter. We do this by making an ansatz with non zero fields living only in an $SO(3)$ subgroup, and then reducing the equations to those of an $SO(3)$ theory coupled to an adjoint of the group. Consider the model Lagrangian:

$$L = -\frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} - (D_\mu \phi)^\dagger (D^\mu \phi)$$

For simplicity we start with $SU(3)$ gauge group and take ϕ as a complex scalar transforming in the fundamental representation. Let H be the real $SO(3)$ subgroup. The fundamental of $SU(3)$ transforms as the adjoint of $SO(3)$, whose Lie algebra we denote by h . We now consider an ansatz with $A_\mu^a = 0$ for $a \in \bar{h}$, the complement of h , and ϕ real. With such an ansatz the field strength is also zero outside h , and the $SU(3)$ covariant derivative reduces to the $SO(3)$ covariant derivative. The field equations are:

$$\begin{aligned} (D^\mu F_{\mu\nu})^a &= i\phi^\dagger T^a D_\nu \phi - i(D_\nu \phi)^\dagger T^a \phi \\ (D^2 \phi)^a &= 0 \end{aligned} \tag{2.1}$$

The scalar field equations and the gauge field equations for $a \in H$ reduce trivially to the corresponding equations for the H gauge theory. It remains to check that the gauge field equations for $a \in \bar{h}$ are also satisfied. This amounts to showing that, for $a \in \bar{h}$:

$$0 = i\phi^t T^a \partial_\nu \phi - i(\partial_\nu \phi)^t T^a \phi + \sum_{b \in h} A_\nu^b \phi^t (T^a T^b + T^b T^a) \phi. \tag{2.2}$$

We have used reality of ϕ to rewrite ϕ^\dagger as ϕ^t , the transpose. The first two terms cancel because, for $a \in \bar{h}$, T^a is symmetric. To see the cancellation of the remaining two terms, use the fact that

$$\{T^a, T^b\} = k\delta^{ab} + d^{abc}T^c.$$

The d^{abc} are real because the T 's are hermitian. For $a \in \bar{h}$, and $b \in h$, T^c is imaginary, and therefore antisymmetric. This guarantees that the last term is zero.

Having reduced the $SU(3)$ equations to $SO(3)$ equations we know that the Prasad-Sommerfield solutions solve them. So there are finite energy, non-singular monopole, dyon, and multimonopole solutions to the equations of $SU(3)$ gauge theory coupled to fundamental matter. These solutions can be embedded in $SU(N)$ coupled to scalars in the fundamental representation. This is done by using an ansatz for $A_\mu^a T^a$ where the only non-zero A_μ^a correspond to generators of an $SU(3)$ subgroup associated with say the first 3×3 block of the $N \times N$ matrices of the fundamental of $SU(N)$. And the only non-zero scalars are taken to live in the first three entries of the N dimensional vector. The reduction of equations of $SU(3)$ to those of $SO(3)$ used the reality of the subgroup, and a similar reduction works for $SU(N)$ and $SO(N)$. Note that the usual embeddings of the BPS solution when the matter is in the adjoint of the original gauge group do not make use of such reality conditions.

The scalar field expectation values at spatial infinity break the gauge symmetry from $SU(N_c)$ to $SU(N_c - 1)$ for the simplest solutions. Since $\pi_2(SU(N_c)/SU(N_c - 1)) \sim \pi_1(SU(N_c - 1))$ is trivial, these solutions are not topologically stable. It has been shown that configurations with the long distance magnetic fields of such monopoles are actually also locally unstable to perturbations of the gauge fields in the unbroken $SU(N_c - 1)$ subgroup [12].

We now describe how to embed these solutions in $N = 1$ supersymmetric gauge theory. Following the conventions of [13], the bosonic terms are:

$$L = -\frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} - (D_\mu \phi)^\dagger (D_\mu \phi) - (\tilde{D}^\mu \tilde{\phi})(\tilde{D}_\mu \tilde{\phi})^\dagger - \frac{e^2}{2} [\phi^\dagger T^a \phi - \tilde{\phi} T^a \tilde{\phi}^\dagger]^2 \quad (2.3)$$

Where ϕ and $\tilde{\phi}$ denote the scalar components of the quark and antiquark superfields respectively. Also, \tilde{D}_μ is the covariant derivative in the complex conjugate fundamental representation of $SU(N)$. Choosing an ansatz with $\phi = \tilde{\phi}^\dagger$ and real, the D -term clearly vanishes, the purely bosonic equations reduce to those of the model Lagrangian considered above, except that we now have two scalar fields. In this way we can carry over the solutions from the model Lagrangian by a simple rescaling. The corresponding Bogomolnyi equations take the form

$$B_i = \pm 4 D_i \phi = \pm 4 (\tilde{D}_i \tilde{\phi})^\dagger \quad (2.4)$$

In writing these equation we made use of the identification between the vector of $SO(3)$ and the fundamental representation of $SU(3)$.

Conditions for the vanishing of D -terms for arbitrary N_f and N_c have been investigated in [14] and yield the well known moduli space of vacua which has been studied classically and quantum mechanically [2]. The simplest solution described above satisfies boundary conditions appropriate for the meson field matrix, $M^i_{\bar{i}}$ acquiring an expectation value of rank one, the case in which we will concentrate in the rest of the paper. However, the construction of solutions can be easily generalised to cases where M has rank smaller than $N_c/3$ by using different $SO(3)$ subgroups of $SU(N_c)$ which couple to different flavours. Note that, at the level of the classical Lagrangian these solutions exist for any N_f . After taking into account quantum corrections [1], these configurations do not have finite energy unless $N_f \geq N_c + 2$ where the classical moduli space is the same as the quantum moduli space.

These solutions break the $N = 1$ supersymmetry. This is clearly seen, for example, from the gluino transformation rules

$$\delta_\xi \lambda^a = i\xi D^a + \sigma^{\mu\nu} \xi F^a_{\mu\nu}. \quad (2.5)$$

due to the non vanishing field strength (the D term was arranged to be zero). The resulting fermionic zero modes in the monopole background lead to states filling out a representation of supersymmetry, as discussed in detail later. In practice, the absence of unbroken supersymmetries means that we do not have much control on the strong coupling physics of these solutions.

3. Embeddings and gauge quantum numbers

The gauge quantum numbers of the monopole can be described in the framework developed by Goddard, Nuyts and Olive (GNO) which has been useful in investigations of Montonen-Olive duality [15] in $N = 4$ theories [16]. In this section we briefly review their construction, discuss the possibilities of some simple variations on their definition of dual group, while keeping the key property that the weights of the dual group are magnetic weights (see definition below) of monopoles. Using these remarks we deduce that the monopoles constructed above transform as adjoints. We will also discuss in some generality the applicability of these definitions to theories with less than $N = 4$ supersymmetry.

The long distance magnetic field defines an element M of the Lie algebra \mathfrak{h} of the unbroken gauge group:

$$F_{ij} = \epsilon_{ijk} \frac{x_k}{x^3} M.$$

By a gauge transformation, $L = eM$ can be taken to lie in the Cartan subalgebra \mathfrak{h} . Using the metric this determines an element \tilde{L} , called a magnetic weight, of the dual vector space \mathfrak{h}^* . L and \tilde{L} are related by $\langle \tilde{L}, h_1 \rangle = (L, h_1)$, for any $h_1 \in \mathfrak{h}$, where the LHS is the evaluation of the functional \tilde{L} on h_1 and (\cdot, \cdot) is the metric on the Cartan. (\cdot, \cdot) induces a metric on \mathfrak{h}^* :

$$(\tilde{L}_1, \tilde{L}_2) = (L_1, L_2).$$

The Dirac quantization condition,

$$e^{4\pi i L} = 1, \tag{3.1}$$

in conventions where $D_\mu = \partial_\mu + ieA_\mu^a T^a$, allows \tilde{L} to live in a lattice, which we will call the Dirac lattice. GNO found that this lattice coincides with the weight lattice of a dual group G^v . The roots of G^v are proportional to the roots of G : $\alpha^v = \frac{\alpha}{(\alpha, \alpha)}$, and the global structure of G^v is determined by

$$G^v = \tilde{G}^v / k(G^v) \tag{3.2}$$

where $k(G^v)$ is the quotient of the weight lattice of G by the root lattice of G , and \tilde{G}^v denotes the universal cover of G^v . Since G is a subgroup of the broken gauge group, \tilde{L} can also be related to an element of the dual of the broken group [17]. In the following discussion we will characterize the monopole in terms of the GNO dual of the unbroken group, but the discussion can be extended to the broken group. We note that the set of $\alpha^v = \frac{k\alpha}{(\alpha, \alpha)}$ defines a root system satisfying the usual axioms for root systems [18], (hence allows reconstruction of a dual group) for any k . Note also that for any k , $(\alpha^v)^v = \alpha$. Dual root systems defined using different values of k are isomorphic to each other as root systems (i.e the isomorphism preserves angles), and hence determine isomorphic Lie algebras.

It is useful to bear in mind that in some cases there may be different ways of defining G^v such that $\Lambda(G^v)$ is the lattice of magnetic weights. Consider the case of $SU(2)$ theories. The basic (spherically symmetric) BPS monopole has a magnetic weight $\frac{\alpha}{(\alpha, \alpha)}$. The prescription of GNO defines the dual group by giving its root as $\frac{\alpha}{(\alpha, \alpha)}$, and its fundamental group as Z_2 . This means that the dual group is $SO(3)$ and the BPS monopole transforms as the adjoint. This is appropriate in $N = 4$ where the spins are consistent with the

monopole being dual to a gauge boson. An alternative way to define a group whose weight lattice is that generated by the basic BPS monopole, is to use $\alpha^v = 2 \frac{\alpha}{(\alpha, \alpha)}$ and declare the fundamental group to be trivial. This picks out G^v as $SU(2)$ and the spherically symmetric BPS monopole transforms as the fundamental representation of this dual group. This is appropriate in interpreting, from the GNO point of view, the self duality of $N = 2$ $SU(2)$ theory with $N_f = 4$ described in [4].

Another simple variation on the GNO construction we will consider is to define the dual group whose weight lattice is a sublattice of the Dirac lattice. We will see that this variation is necessary in order to have a simple assignment of gauge quantum numbers for the monopoles of section two. It will also allow us to address a puzzle which appears if one tries to understand some aspects of the $N = 1$ duality of [1] using the GNO construction. A direct understanding of the duality between $SU(N_c)$ and $SU(N_f - N_c)$ appears hard, so one might hope to understand the duality at the point $N_f = 2N_c$ in terms of the GNO type of construction and relate the other cases to these self-dual points using flows described in [1] (deriving the general duality from the self dual case has been discussed in the context of relations with $N = 2$ in [1,19]). The puzzle is that even at the self-dual point, the standard GNO definition gives the dual group of $SU(N_c)$ as $SU(N_c)/Z_{N_c}$ which does not have fundamental representations. On the other hand both the electric and magnetic theories in [1] have fundamental quarks. We will show that considering sublattices of the Dirac lattice suggests a possible resolution of this puzzle.

We now characterize the gauge quantum numbers of the monopoles constructed in section 2 using the formalism of GNO. For simplicity consider $SU(N_c)$ with $N_f = 1$ in a vacuum where $\langle \phi_1 \rangle = \langle \tilde{\phi}_1 \rangle \neq 0$. A simple class of embeddings of $SO(3)$ generalizing the one discussed for $SU(3)$ allows us to construct solutions for this vacuum. These are characterized by three distinct integers $(1jk)$ picked from 1 to N_c . For each choice $(1jk)$ the non-zero components of ϕ for the solution written in 't Hooft Polyakov form [6], are ϕ_1 , ϕ_j and ϕ_k . Rotating to a gauge where ϕ points in a fixed direction [20], we arrange for the only non-zero component to be ϕ_1 . The long distance field in such a gauge is $F_{ab} = \pm i[E_{jk} - E_{kj}] \epsilon_{abc} \frac{x_c}{ex^3}$, the matrix in the chosen $SO(3)$ which annihilates ϕ_1 . In this way we can construct $(N_c - 1)(N_c - 2)$ monopole solutions.

We conjugate $\pm[E_{jk} - E_{kj}]$ by $e^{\frac{i\pi}{4}(E_{jk} + E_{kj})}$ to get the set of matrices $\pm(H_j - H_k)$ where H_j is the matrix with 1 in the j 'th diagonal and 0 everywhere else. So we have a set of $(N_c - 1)(N_c - 2)$ magnetic weights. These are all related by $W(G)$, the Weyl

group of G , and are gauge equivalent in G . Now if $\alpha^v = \frac{k\alpha}{(\alpha, \alpha)}$, and $W_\alpha(\beta)$ is the Weyl reflection of β in the hyperplane perpendicular to the root α , $W_\alpha(\beta) = (W_{\alpha^v}(\beta^v))^v$. So the magnetic weights lie in one orbit of $W(G^v)$. In addition, for each magnetic weight in the set, there is also the negative of the weight present in the set. We want to recognise this set of weights of the GNO dual group, as the non-zero weights of some representation of G^v . The behaviour of the set under $W(G^v)$ rules out reps like the fundamental which are not self-conjugate.

The roots of $SU(N_c - 1)$ take the form $L_i - L_j$ (i and j are distinct integers between 1 and $N_c - 1$), where the L_i are linear functionals on the Cartan subalgebra of $SU(N_c - 1)$ defined by $\langle L_i, \sum a_j H_j \rangle = a_i$, and $\sum a_j H_j$ is any traceless diagonal matrix. Pick a metric on the Lie algebra, some arbitrary constant c times trace in the fundamental. With this metric, if $L = H_j - H_k$, we have $\tilde{L} = c(L_j - L_k)$, since $c \operatorname{tr}(L \sum a_j H_j) = \langle \tilde{L}, \sum a_j H_j \rangle$. Now if $\alpha = L_i - L_j$, the corresponding element of \mathfrak{h} is $\frac{1}{c}(H_i - H_j)$, so $(\alpha, \alpha) = c \operatorname{tr} \frac{(H_i - H_j)^2}{c^2} = 2/c$. Therefore $\frac{\alpha}{(\alpha, \alpha)} = \frac{c}{2}(L_i - L_j)$. It follows that the magnetic weights \tilde{L} of the monopole are twice the roots of the GNO dual group. (For comparison, note that the BPS monopole as a solution to $SU(2)$ gauge theory with adjoint Higgs defines a magnetic weight equal to $\frac{\alpha}{(\alpha, \alpha)}$ and has the minimal Dirac unit of charge according to definition (3.1)). The smallest representation containing these weights is associated with a Young diagram having two columns of length $N_c - 2$ and two columns of length one. However such a representation has other non-zero weights as well, which cannot be extracted from the monopoles, so assigning the monopoles to such representations is problematic.

Instead define the roots of the dual group by $\alpha^v = \frac{2\alpha}{(\alpha, \alpha)}$. And fix the global structure of the group by identifying the root and weight lattices, so that the dual group is $SU(N_c - 1)/Z_{N_c - 1}$. Then the magnetic weights of the monopoles we have constructed are equal to the roots of the dual group, and hence correspond to the adjoint of the dual group. With this definition of dual group, the $(N_c - 1)(N_c - 2)$ magnetic weights of the monopole exhaust the set of non-zero weights of the adjoint representation. We may physically motivate this definition of the dual group. It appears likely that not all magnetic weights allowed by Dirac quantization in $SU(N_c)$ theories coupled to fundamental matter correspond to non-singular finite energy monopoles, and the only such monopoles might be multimonopoles based on those we have constructed (with magnetic weights being integer linear combinations of those we have described). If this is indeed true, then the dual group defined above is

distinguished in that its weight lattice is the set of magnetic weights of non-singular, finite energy, classical monopoles in super QCD.

The following remark is not directly relevant to the interpretation of the monopoles we are studying in this paper, but is suggested by the above discussion. It may be useful in trying to relate monopoles to duality in $N = 1$, to explore the possibility of constructing the dual group, using some sublattice of the Dirac lattice, not necessarily the sublattice we chose above. If we use the Dirac lattice, the dual group of $SU(N)$ is $SU(N)/Z_N$, which does not have fundamental representations (whereas in $N = 1$ duality both electric and magnetic sides can have fundamental quarks). However the weight lattice of $SU(N)/Z_N$ contains sublattices isomorphic, by a rescaling, to an $SU(N)$ weight lattice. To see an example, note that the weight lattice of $SU(N)/Z_N$ is generated by $L_i - L_j$ with the relation $\sum L_i = 0$ (see for example [21]). In this lattice $NL_1 = (L_1 - L_2) + (L_1 - L_3) + \cdots (L_1 - L_N)$. The NL_i generate a sublattice with the relation $\sum NL_i = 0$, which is isomorphic by a rescaling to an $SU(N)$ weight lattice. This suggests a definition of the GNO type dual group of $G = SU(N)$ by the prescription $\alpha^v = \frac{N\alpha}{(\alpha, \alpha)}$, and G^v simply connected. With this prescription G^v is isomorphic as a group to $SU(N)$, and its weight lattice is a sublattice of the Dirac Lattice. So the dual group defined using such a sublattice might be more closely related to the $N = 1$ dual group at the self dual points, e.g $N_f = 2N_c$ of [1]. Restricting to this sublattice would have to be motivated by some extra physical input in addition to the Dirac condition (e.g stability). It is extremely interesting that the weights of the fundamental and antifundamental representations of the dual group (to $SU(N)$) defined in this way are actually the magnetic weights of spherically symmetric monopoles constructed as solutions to $SU(N)$ gauge theory in [22]. These monopoles are constructed using the maximal embedding of $SO(3)$ in $SU(N)$, and live in a phase where $SU(N)$ is broken to $SU(N-1) \times U(1)$. These monopoles require adjoint matter for their construction, and will not be discussed further in this paper.

4. Zero mode quantum mechanics

The bosonic solution described in the previous section can be acted on by the symmetries it breaks to generate other solutions of the same energy. For example when $N_f > 1$ one can start with a solution where M_1^1 tends to a non-zero constant at infinity, and rotate by $SU(N_f)$ matrices to get a family of solutions parametrized by

$(SU(N_f) \times SU(N_f))/(SU(N_f-1) \times SU(N_f-1))$. The monopoles are $SU(N_f-1)$ singlets: but this is consistent with it being a fundamental or a singlet of $SU(N_f) \times SU(N_f)$. Even if the vacuum order parameter is in the fundamental of $SU(N_f) \times SU(N_f)$, the monopole can be a singlet. A familiar example is the $N = 4$ theory, where the $O(6)$ symmetry is spontaneously broken, and the monopole, being dual to the W boson, is interpreted as a singlet of $O(6)$. One can also use the $U(1)$ symmetries to rotate ϕ and $\tilde{\phi}$ while leaving their magnitude constant. However, in the monopole solution, these symmetries are spontaneously broken by the vacuum boundary conditions of the scalar fields at spatial infinity. Thus these rotations do not give rise to normalisable zero modes, or excitations of finite charge, so they will not be quantized in the following discussion of the quantum numbers of the monopole.

The low energy dynamics of the monopoles described in the previous section is dominated by their unstable character. At best, they can be quantized as unstable resonances, possibly with an interpretation similar to sphalerons. However, their quantum numbers in a single particle Hilbert space are still determined by the effective quantum mechanics of the zero modes around the classical solution. For similar recent discussions in $N = 2$ theories see [23,24,4,25,26,27]. In this paper we will only consider the single monopole sector, where one finds the usual collective coordinates associated with space translations and charge rotations, leading to dyonic excitations of the monopole, as well as fermionic zero modes, whose quantization leads to non trivial spin degrees of freedom. The corresponding fermionic equations of motion are given by

$$\begin{aligned}\bar{\sigma}^\mu D_\mu \lambda^a &= \sqrt{2}e \sum_{r=1}^{N_f} (\bar{\psi}_r T^a \phi_r + \tilde{\phi}_r T^a \bar{\tilde{\psi}}_r) \\ \bar{\sigma}^\mu D_\mu \psi_r &= -\sqrt{2}e \bar{\lambda}^a T^a \phi_r \\ \bar{\sigma}^\mu \tilde{D}_\mu \tilde{\psi}_r &= -\sqrt{2}e \bar{\lambda}^a \tilde{\phi}_r T^a\end{aligned}\tag{4.1}$$

Since the PS monopole breaks all the supersymmetries, an obvious solution of these equations is given by the supersymmetry variation of the bosonic PS solution¹. We find a

¹ Note that, unlike the flavor symmetries, the supersymmetry is *not* spontaneously broken by the vacuum boundary conditions at infinity. Supersymmetry is asymptotically restored at long distances, and it makes sense to quantize these zero modes as collective coordinates.

four parameter family of zero modes

$$\begin{aligned}
\lambda_\epsilon &= -\frac{i}{\sqrt{2M}} B_{PS}^i \sigma_i \epsilon \\
\psi_\epsilon &= \frac{i}{\sqrt{M}} D_i \phi_{PS} \sigma^i \bar{\epsilon} \\
\tilde{\psi}_\epsilon &= \frac{i}{\sqrt{M}} \tilde{D}_i \tilde{\phi}_{PS} \sigma^i \bar{\epsilon}
\end{aligned} \tag{4.2}$$

where ϵ is related to the anticommuting parameter ξ of (2.5) by $\epsilon = \frac{\xi}{\sqrt{2M}}$. These modes solve (4.1) with non trivial Yukawa terms along the $r = 1$ flavour direction, since $\phi_{PS} = \tilde{\phi}_{PS}^\dagger$ vanish for $r \neq 1$. The remaining equations for the $r > 1$ flavours are

$$\bar{\sigma}^\mu D_\mu \psi_r = \bar{\sigma}^\mu \tilde{D}_\mu \tilde{\psi}_r = 0 \tag{4.3}$$

and have no normalizable solutions in the PS monopole background. To see this, let us assemble ψ_r and $\tilde{\psi}_r$ into a Dirac spinor Ψ_r . Then, eq. (4.3) is the $G = 0$ case of

$$i\gamma^\mu D_\mu \Psi_r + iG[\phi_{PS}, \Psi_r] = 0 \tag{4.4}$$

where the commutator is to be regarded as the adjoint action of $SU(2)$. This is consistent because the PS solution is in the vector representation of the distinguished $SO(3)$ where the monopole sits, and this is equivalent to the adjoint action in (4.4). The particular case $G = 1$ is well known as it represents the zero mode equation for the fermionic component of a $N = 2$ vector multiplet. Also, if we define $A_0 \sim \phi_{PS}$ we have the zero mode equation for adjoint fermions in a particular four dimensional instanton background.

The analysis of Jackiw and Rebbi [28] shows that (4.4) has two chiral normalizable zero modes along each flavour direction as long as the adjoint Yukawa term is non zero, $G \neq 0$. A simple inspection of their results reveals that both zero modes become non normalizable as $G \rightarrow 0$. For the PS monopole configuration, normalizing the scalar expectation values $\langle \phi \rangle = 1$, the solutions have the form

$$\Psi^a \sim [f_1(r) \hat{r}^a \sigma_i \hat{r}^i + f_2(r) (\sigma^a - \hat{r}^a \sigma_i \hat{r}^i)] \chi$$

with χ a Weyl spinor and a the $SU(2)$ adjoint index. The radial functions $f_1(r)$ and $f_2(r)$ must be normalizable in the norm $\|f_i\| = \int |f_i|^2 r^2 dr$ and are related via

$$f_1(r) = \sinh r \left(f_2'(r) + \frac{1-G}{r} f_2(r) \right) + G \cosh r f_2(r) \tag{4.5}$$

In terms of the function $u(r)$ defined as

$$f_2(r) = r^{\frac{G-3}{2}} (\sinh r)^{-\frac{1-G}{2}} u(r) \quad (4.6)$$

the zero mode equation reduces to the following zero energy Schrödinger problem:

$$-u''(r) + V_{\text{eff}}(r)u = 0 \quad (4.7)$$

The effective potential V_{eff} is a monotonic decreasing function with asymptotics $V(r \rightarrow 0) \sim 2/r^2$ and $V(r \rightarrow +\infty) \sim (1-G)^2/4 + O(1/r)$. There is one regular solution at the origin, as well as a singular one $f_2(r \rightarrow 0) \sim c_1 + c_2 r^{-3}$. The long distance behaviour is $f_2(r \rightarrow \infty) \sim c_1' r^{G-1} e^{-Gr} + c_2' r^{-2} e^{-r}$. The corresponding behaviour of f_1 is $f_1(r \rightarrow \infty) \sim d_1 r^{G-1} e^{-(G+1)r} + d_2 r^{-2}$, where d_1 is determined in terms of c_1' and d_2 is determined in terms of c_2' . The c_1' component comes from the exponentially increasing solution of (4.7), $u \sim e^{\frac{1-G}{2}r}$ and is non normalizable for $G = 0$. For non-zero G , both solutions for f_1 and f_2 are well behaved at infinity, so in particular the solution regular at the origin is normalizable. For $G = 0$ only one solution is normalizable at infinity. Although f_1 is always well behaved at long distances, f_2 is only normalizable for the choice $c_1' = 0$. Thus an acceptable solution to the system can only exist if the solution regular at the origin happens to be the one regular at infinity. But if such a regular f_2 existed, the corresponding $u(r)$ obtained from (4.6) would also be normalizable. The behaviour of the effective potential in (4.7) rules this out. Indeed, since $V_{\text{eff}}(r) > (1-G)^2/4$ we cannot have a zero energy solution of (4.7) which is regular both at the origin and at infinity. This means that, if $c_2 = 0$, then $c_1' \neq 0$ in the $G \rightarrow 0$ limit and $f_2(r)$ is not normalizable.

As a consequence, it seems that the only normalizable fermionic zero modes in the single monopole sector are those generated by supersymmetry. This circumstance greatly simplifies the quantization of the associated collective coordinates, since explicit fermionic solutions of the equations of motion can be written down by simple supersymmetry rotations of the original bosonic solution. To second order in the fermionic parameter ϵ we find

$$\begin{aligned} A_\epsilon^i &= A_{PS}^i + \frac{1}{2M} \bar{\epsilon} B_{PS}^i \epsilon \\ A_\epsilon^0 &= \frac{1}{2M} \bar{\epsilon} \sigma^i B_{PS}^i \epsilon \\ \phi_\epsilon &= \phi_{PS} + \frac{i}{2M} \bar{\epsilon} \sigma^i D_i \phi_{PS} \epsilon \\ \tilde{\phi}_\epsilon &= \tilde{\phi}_{PS} + \frac{i}{2M} \bar{\epsilon} \sigma^i \tilde{D}_i \tilde{\phi}_{PS} \epsilon \end{aligned} \quad (4.8)$$

This solution, together with (4.2), can be made into a collective coordinate ansatz by simply giving time dependence to ϵ . Upon direct substitution into the field theory action we obtain the following effective lagrangian for the fermionic coordinates:

$$\mathcal{L}_{\text{eff}} = i \bar{\epsilon} \partial_t \epsilon \quad (4.9)$$

where $\bar{\epsilon} \epsilon = \bar{\epsilon}_{\dot{\alpha}} \delta_{\dot{\alpha}\alpha} \epsilon_{\alpha}$. Canonical quantization of this fermionic quantum mechanics leads to

$$\{\epsilon_{\alpha}, \bar{\epsilon}_{\dot{\alpha}}\} = \delta_{\alpha\dot{\alpha}} \quad (4.10)$$

so that $\bar{\epsilon}_{\dot{\alpha}}$, $\dot{\alpha} = 1, 2$ generates a Fock space of four states. Since ϵ was the supersymmetry parameter, if the vacuum is regarded as a scalar (corresponding to the spherically symmetric PS solution), then we have two spin zero states and two polarizations of spin 1/2, a standard massive representation of $N = 1$ supersymmetry. The spin operator acts in the fermionic quantum mechanics as $J_3 = \frac{1}{2} \bar{\epsilon} \sigma_3 \epsilon$, and satisfies $[J_3, \epsilon_{\alpha}] = -\frac{1}{2} (\sigma_3 \epsilon)_{\alpha}$. Altogether, we find the degrees of freedom of a massive chiral superfield.

The bosonic collective coordinates are easily introduced by giving an implicit time dependence to all bosonic fields in terms of the position collective coordinate $X^i(t)$. In practice, this amounts to the following rule for time derivatives:

$$\partial_t(\text{Boson}) = \dot{X}^i \partial_i(\text{Boson})$$

We can also consider $U(1)$ charge rotations at infinity by an angle $\chi \in [0, 2\pi)$, leading to dyon solutions, which simply modify the ansatz (4.8) by the addition of a term $\dot{\chi} \phi_{PS}$ to A_0 . Finally, the ansatz for A_0 should be further corrected in order to satisfy Gauss law, with the final result:

$$A_0(\epsilon, \chi, X^i) = -\frac{1}{2M} \bar{\epsilon} \sigma^i B_{PS}^i \epsilon + \dot{\chi} \phi_{PS} + \dot{X}^i A_{PS}^i \quad (4.11)$$

where we have retained only terms of first order in time derivatives, and second order in anticommuting parameters. This is enough to satisfy the field equations to this order, and further ensure that the collective motion is orthogonal to gauge transformations. Plugging the complete ansatz into the field theory action, and integrating over space we arrive at the complete effective quantum mechanics in the one-monopole sector

$$\mathcal{L}_{\text{eff}} = \frac{M}{2} \dot{X}^i \dot{X}^i + \frac{M}{8} \dot{\chi}^2 + i \bar{\epsilon} \dot{\epsilon} \quad (4.12)$$

where we have used the BPS relations for the monopole mass: $M = \int d^3x (B_{PS})^2 = 4 \int d^3x (D\phi_{PS})^2$. After quantization, the conjugate of X^i becomes the standard space momentum, while the conjugate of χ , a compact coordinate, becomes the quantized electric charge of the dyon.

5. Global quantum numbers

The effective fermionic quantum mechanics is useful to calculate low energy dynamics of monopoles and dyons. Even in the single monopole sector it gives important information, like the quantum numbers of the soliton with respect to the unbroken global symmetries. In this case the unbroken flavour group $SU(N_f - 1)_L \times SU(N_f - 1)_R$ and two $U(1)$ symmetries: the unbroken baryon number and R-symmetry. We simply need to represent the corresponding charges in the quantum mechanical Fock space. Following ref. [28], this is easily accomplished by direct reduction of the field theory Noether currents in the collective coordinate ansatz. For example, the unbroken flavour currents in the field theory are

$$j_R^{a\mu} = \frac{1}{2} \sum_{r,s=2}^{N_f} \left(\bar{\psi}^r (t^a)_r^s \bar{\sigma}^\mu \psi_s - \psi_r (t^a)_r^s \sigma^\mu \bar{\psi}^r \right) + \text{bosons}$$

$$j_L^{a\mu} = \frac{1}{2} \sum_{r,s=2}^{N_f} \left(\bar{\tilde{\psi}}_s (t^a)_r^s \bar{\sigma}^\mu \tilde{\psi}^r - \tilde{\psi}^r (t^a)_r^s \sigma^\mu \bar{\tilde{\psi}}_s \right) + \text{bosons}$$

Since the fermionic zero modes have no component along the $r = 2, 3, \dots, N_f$ flavours, the currents act trivially on the monopole chiral superfield: it is a singlet of $SU(N_f - 1)_L \times SU(N_f - 1)_R$. In a similar fashion, the general $U(1)$ current is given by

$$j^\mu = \frac{q_\lambda}{2} \left(\bar{\lambda}^a \bar{\sigma}^\mu \lambda^a - \lambda^a \sigma^\mu \bar{\lambda}^a \right) + \sum_{r=1}^{N_f} \frac{q_\psi^r}{2} \left(\bar{\psi}^r \bar{\sigma}^\mu \psi_r - \psi_r \sigma^\mu \bar{\psi}^r \right) + \frac{\tilde{q}_\psi^r}{2} \left(\bar{\tilde{\psi}}_r \bar{\sigma}^\mu \tilde{\psi}^r - \tilde{\psi}^r \sigma^\mu \bar{\tilde{\psi}}_r \right) \\ - i q_\phi^r \left((D^\mu \phi^\dagger)^r \phi_r - \phi^{\dagger r} D^\mu \phi_r \right) - i \tilde{q}_\phi^r \left(\tilde{\phi}^r \tilde{D}^\mu \tilde{\phi}_r^\dagger - (\tilde{D}^\mu \tilde{\phi})^r \tilde{\phi}_r^\dagger \right) \quad (5.1)$$

where all the currents are defined as normal ordered with respect to the perturbative vacuum. The original baryon number symmetry with charges $q_\lambda = 0$, $q_\psi^r = q_\phi^r = -\tilde{q}_\psi^r = -\tilde{q}_\phi^r = 1$ is broken in the monopole configuration by the squark expectation values $\langle \phi_1 \rangle = \langle \tilde{\phi}_1 \rangle \neq 0$. One can define an unbroken $U(1)_{B'}$ by combining the previous one with a vector $SU(N_f)$ transformation such that ϕ_1 and $\tilde{\phi}_1$ remain invariant. The corresponding charge assignments are $(q')^1 = -(\tilde{q}')^1 = 0$ and $q' = -\tilde{q}' = \frac{N_f}{N_f - 1}$ for the rest of the flavours. Again, since the supersymmetry zero modes are non zero only along the $r = 1$ flavour component, the new currents act trivially and the monopole has baryon number zero. The situation is different for the R-symmetry. The original anomaly free R-symmetry has charges

$$q_\lambda = 1 \quad , \quad q_\phi^r = \tilde{q}_\phi^r = \frac{N_f - N_c}{N_f}$$

with the usual grading $q_\psi = q_\phi - 1$ within the chiral supermultiplet. The $r = 1$ squark expectation values break this symmetry, which nevertheless can be combined with an axial $SU(N_f)$ transformation to define an unbroken $U(1)_{R'}$ with charges $q'_\lambda = 1$, $(q')^1 = (\tilde{q}')^1 = 0$ and $q' = \tilde{q}' = \frac{N_f - N_c}{N_f - 1}$ for the rest of the flavours. Note that this R' symmetry is non-anomalous with the new massless field content left after Higgs mechanism. The charge operator acting on the fermionic Fock space can be readily calculated as

$$Q_{R'} = \frac{1}{2} \int d^3x \left(\bar{\lambda}_\epsilon^a \lambda_\epsilon^a - \lambda_\epsilon^a \bar{\lambda}_\epsilon^a \right) - \frac{1}{2} \int d^3x \left(\bar{\psi}_\epsilon \psi_\epsilon - \psi_\epsilon \bar{\psi}_\epsilon + \bar{\tilde{\psi}}_\epsilon \tilde{\psi}_\epsilon - \tilde{\psi}_\epsilon \bar{\tilde{\psi}}_\epsilon \right) = \bar{\epsilon} \epsilon - 1 \quad (5.2)$$

The R charge of the monopole vacuum is -1 , and the grading within the massive supermultiplet is given by

$$[Q_{R'}, \epsilon_\alpha] = -\epsilon_\alpha \quad , \quad [Q_{R'}, \bar{\epsilon}_{\dot{\alpha}}] = \bar{\epsilon}_{\dot{\alpha}}$$

We conclude that the monopole chiral superfield has R charge one, compatible with a usual mass term in a dual effective lagrangian description of the form $W_m \sim mY^2$.

6. Discussion

At present, we do not have a concrete proposal regarding the role of these solutions in the general picture of duality. We make some brief comments on the possibility of seeing semiclassical signals of $N = 1$ duality in the sense of finding states which resemble the dual magnetic degrees of freedom. The semiclassical analysis of monopoles looks at an object whose mass increases at large Higgs vev, and for asymptotically free theories the semiclassical methods become more reliable for large vevs. On the other hand $N = 1$ duality is a statement about the theory in the far infrared. So there are two possible scenarios in which one can compare the semiclassical calculations to duality. One is to compare with possible extensions of duality beyond the far infrared, and the other is to consider the behaviour of the states constructed in the semiclassical quantisation as the Higgs vev is tuned to zero, and we flow towards the origin of moduli space. Because of the lack of stability and BPS saturation property the semiclassical monopoles are not guaranteed to define states which survive in the strong coupling region. Since it is not impossible that they do survive and become stable at strong coupling by a currently unknown mechanism, we compare the monopoles constructed here with some of the characters in $N = 1$ duality.

The simplest class of models to consider is $SU(N_c)$ with N_f flavours, with $N_c + 2 \leq N_f \leq 3N_c$. The spin degrees of freedom and the unbroken $U(1)_{R'}$ charge are the same as those of the dual quark which acquires a mass when the electric squark acquires an expectation value. But the baryon number of the heavy dual quarks under $U(1)_{B'}$ is given by $\frac{N_f}{N_f - N_c}$, while we found $B' = 0$ for the monopole. Another difference is that these quarks come in quark-antiquark pairs related by charge conjugation, whereas the monopole is self conjugate. Solutions with positive or negative magnetic charge are gauge equivalent.

These monopoles can be embedded in the theories considered in [29], in which the electric theory has an adjoint X and the magnetic theory has an adjoint Y , in addition to sets of quarks and antiquarks. The steps in the semiclassical treatment are very similar. The monopoles embedded in the electric theory have many of the properties of the Y field. The Y are adjoint chiral superfields, have baryon number zero and are singlets under the flavour group. However two problems remain in identifying the monopoles as candidates for Y . The R-charge of Y under the unbroken $U(1)_{R'}$ is not 1. In characterizing the gauge quantum numbers of these monopoles, the dual groups of $SU(N)$ groups have the form $SU(N)/Z_N$, which are different from the dual groups entering $N = 1$ duality of [1] or [29] even at the self-dual points (for more details on this issue see section 3).

In summary, the physical interpretation of these solutions is not straightforward. However a simple characterization of their transformation properties under Lorentz, global symmetry and appropriate dual gauge groups is possible. It would therefore be interesting to search for consistent scenarios relating these and other monopoles in $N = 1$ theories to duality, perhaps by embedding them in theories which allow stability and yet are related to super-QCD by some simple perturbation. It is also possible that the unstable monopoles described here could find some application in other contexts. For example, unstable monopoles in QCD were proposed in [30] as a heuristic mechanism for the generation of “magnetic mass” in high temperature QCD.

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